The Domino Tiling Problem

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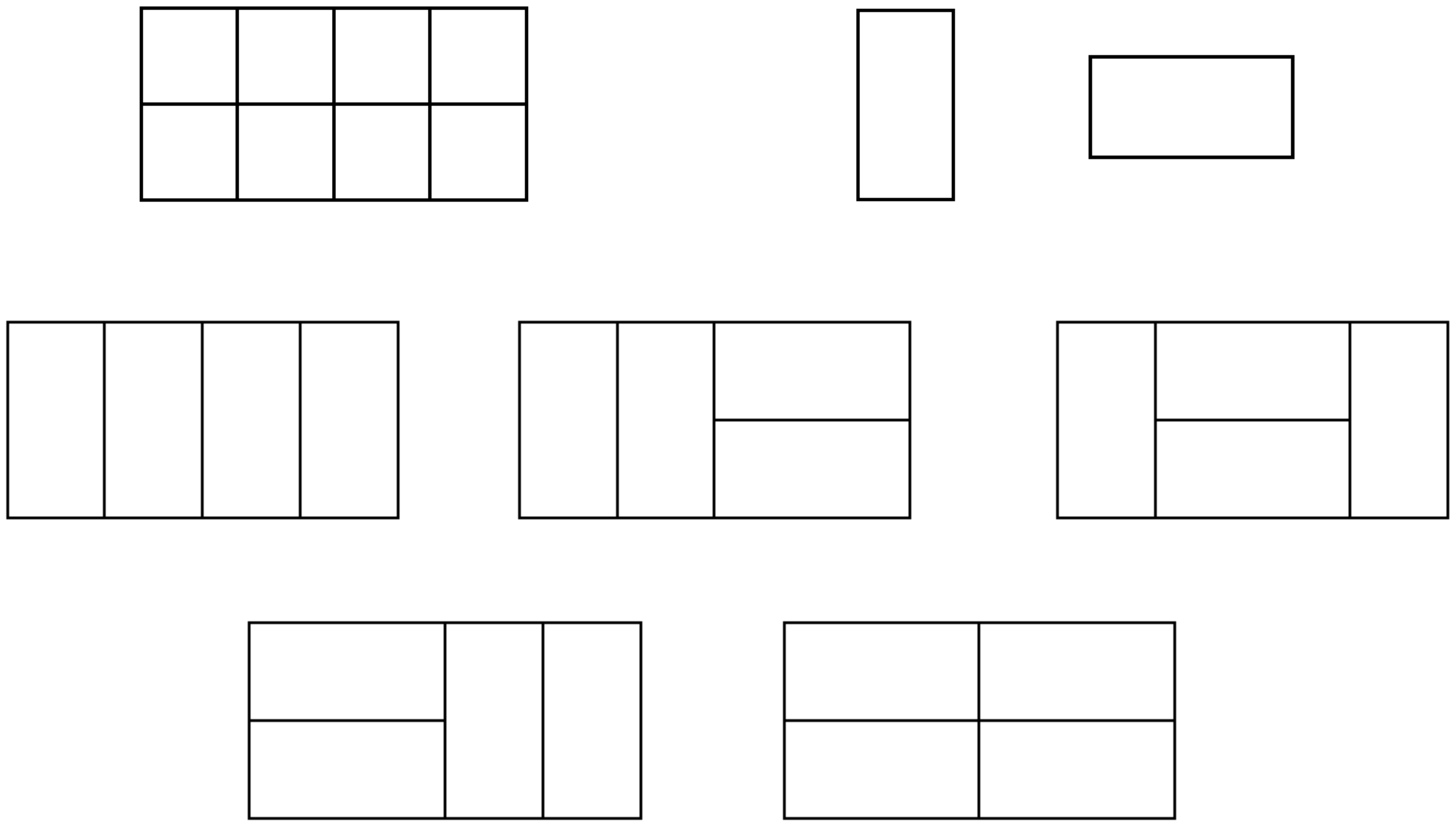
What is a Domino Tiling?

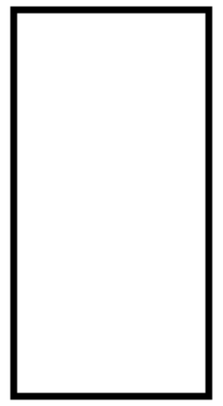


How many domino tilings are there for an n x m grid?

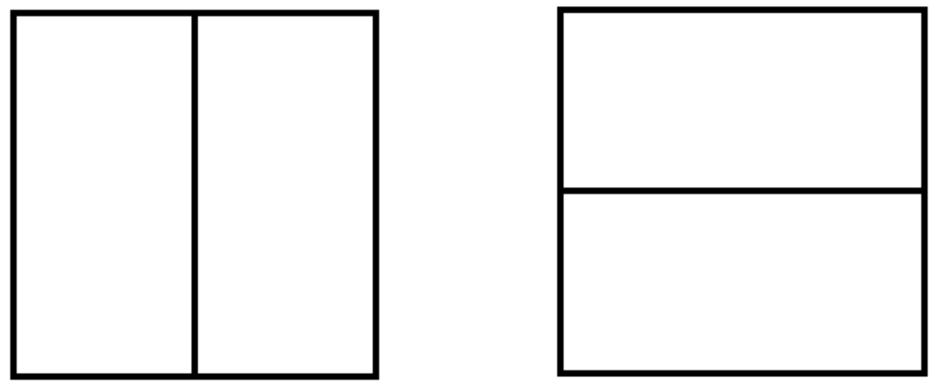
$T_{n,m}$ = # of domino tilings of an n x m grid

When n=2

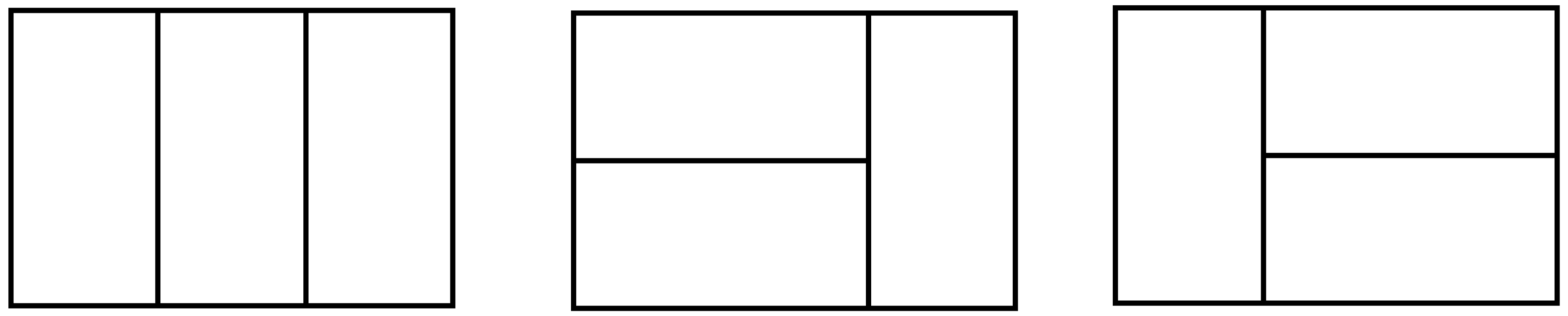
2x1:



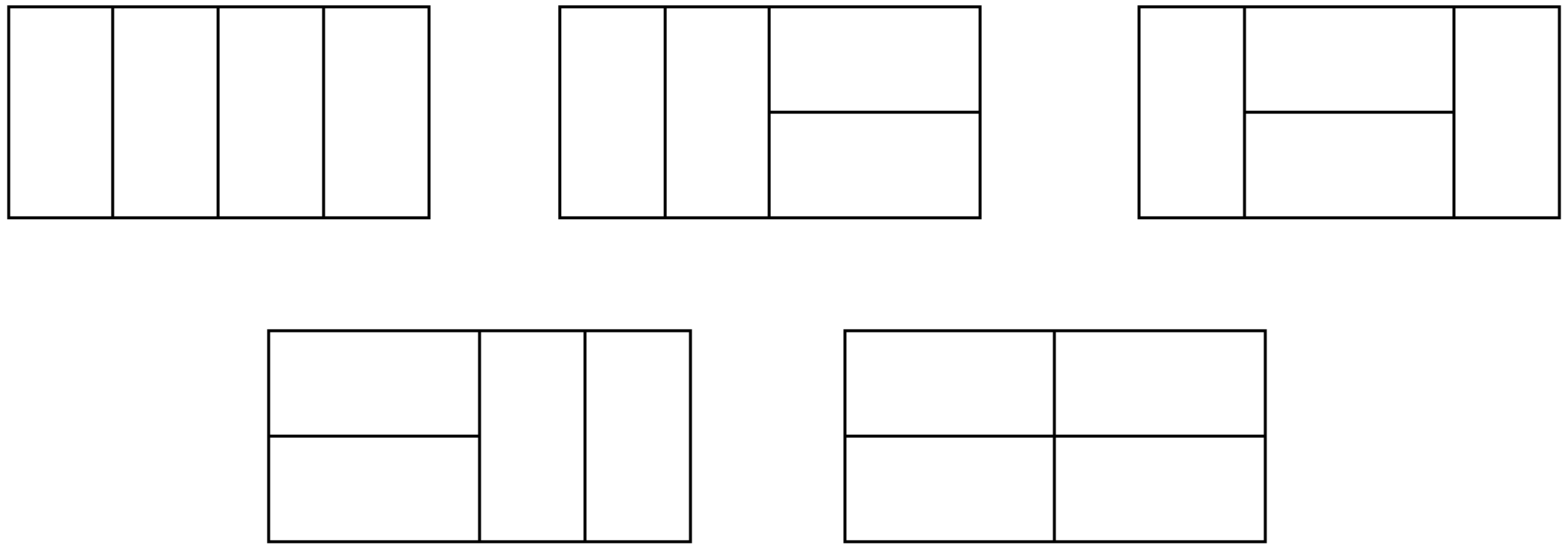
2x2:



2x3:



2x4:



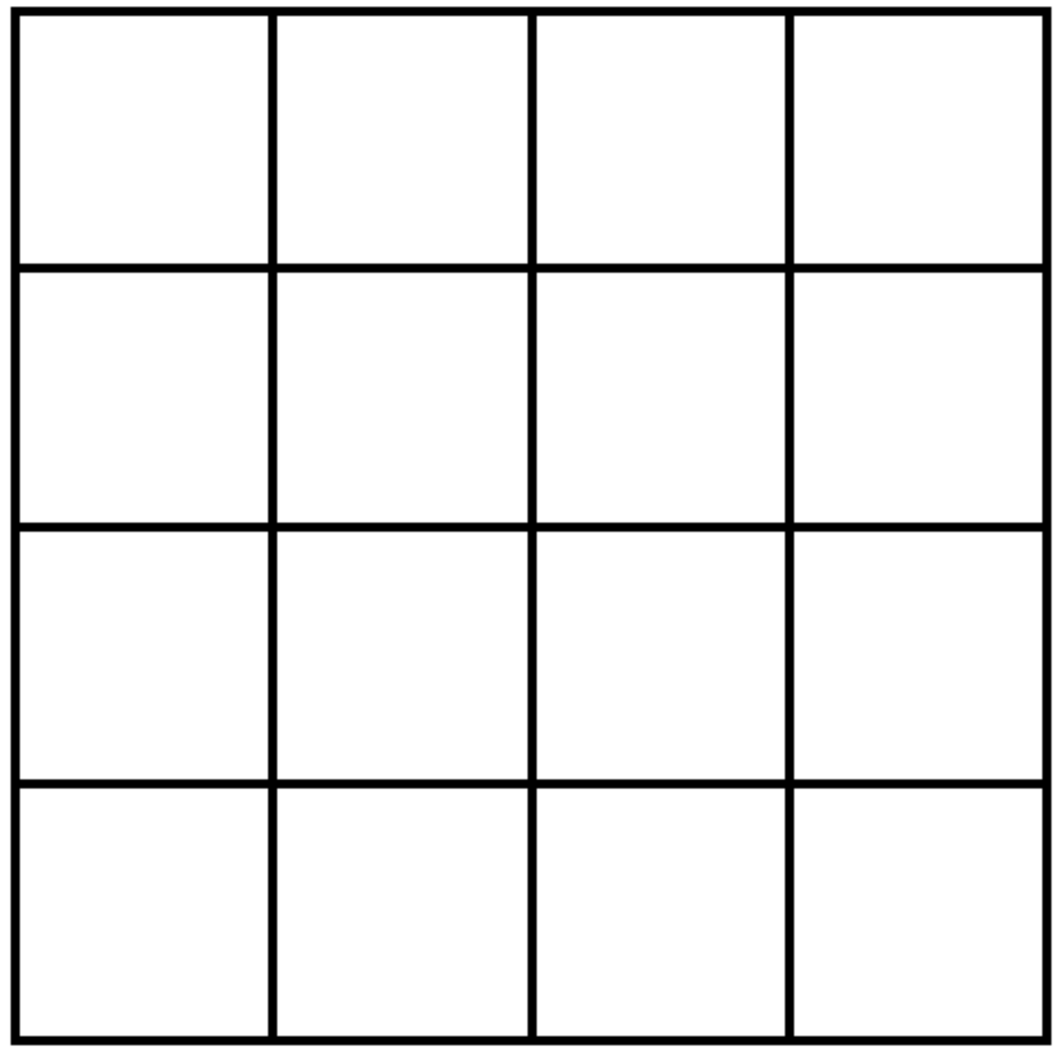
When n=2

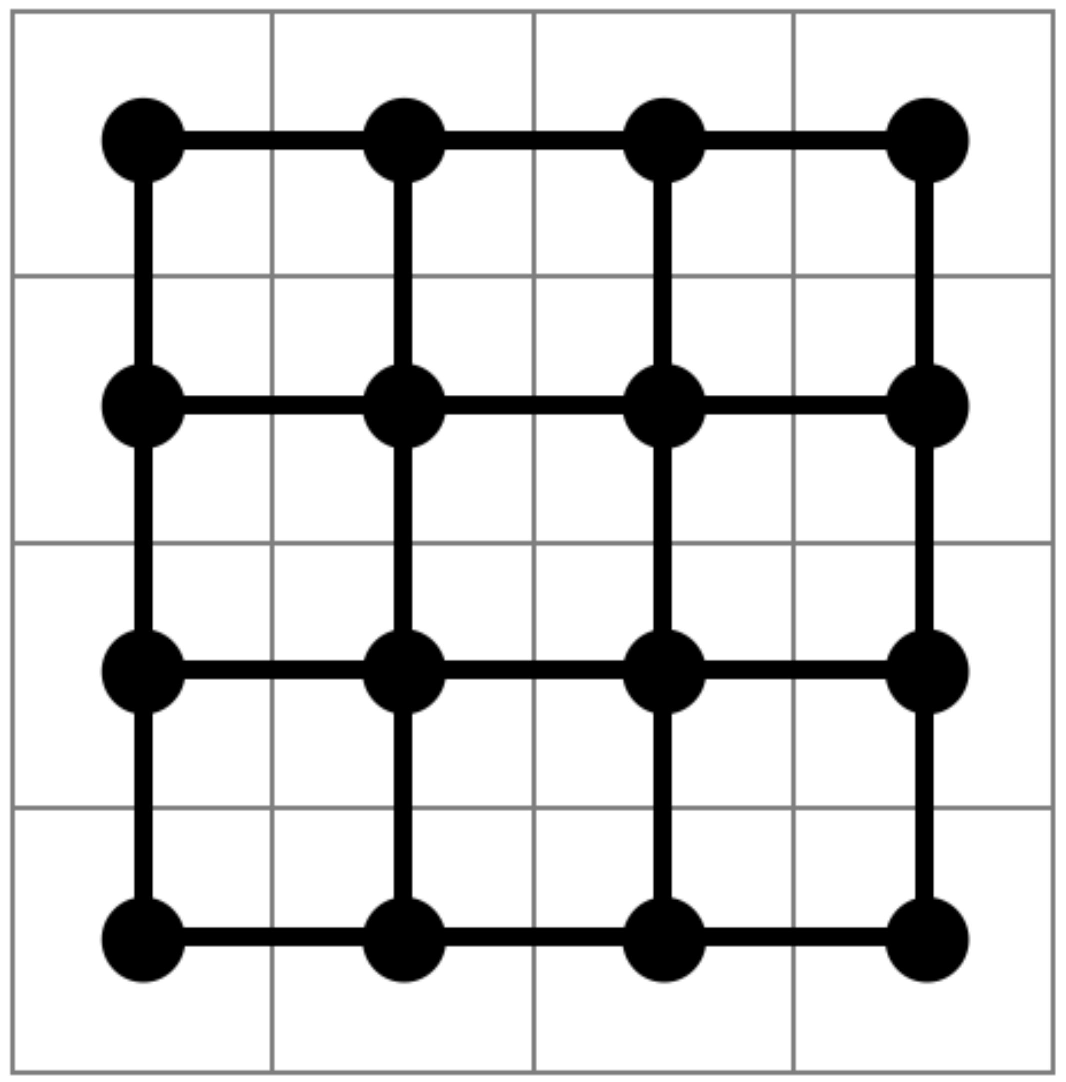
- 2x1: 1
- 2x2: 2
- 2x3: 3
- 2x4: 5
- 2x5: 8
- 2x6: 13

$T_{2,m}$ = The m^{th} Fibonacci number

Translating the Problem into Graph Theory

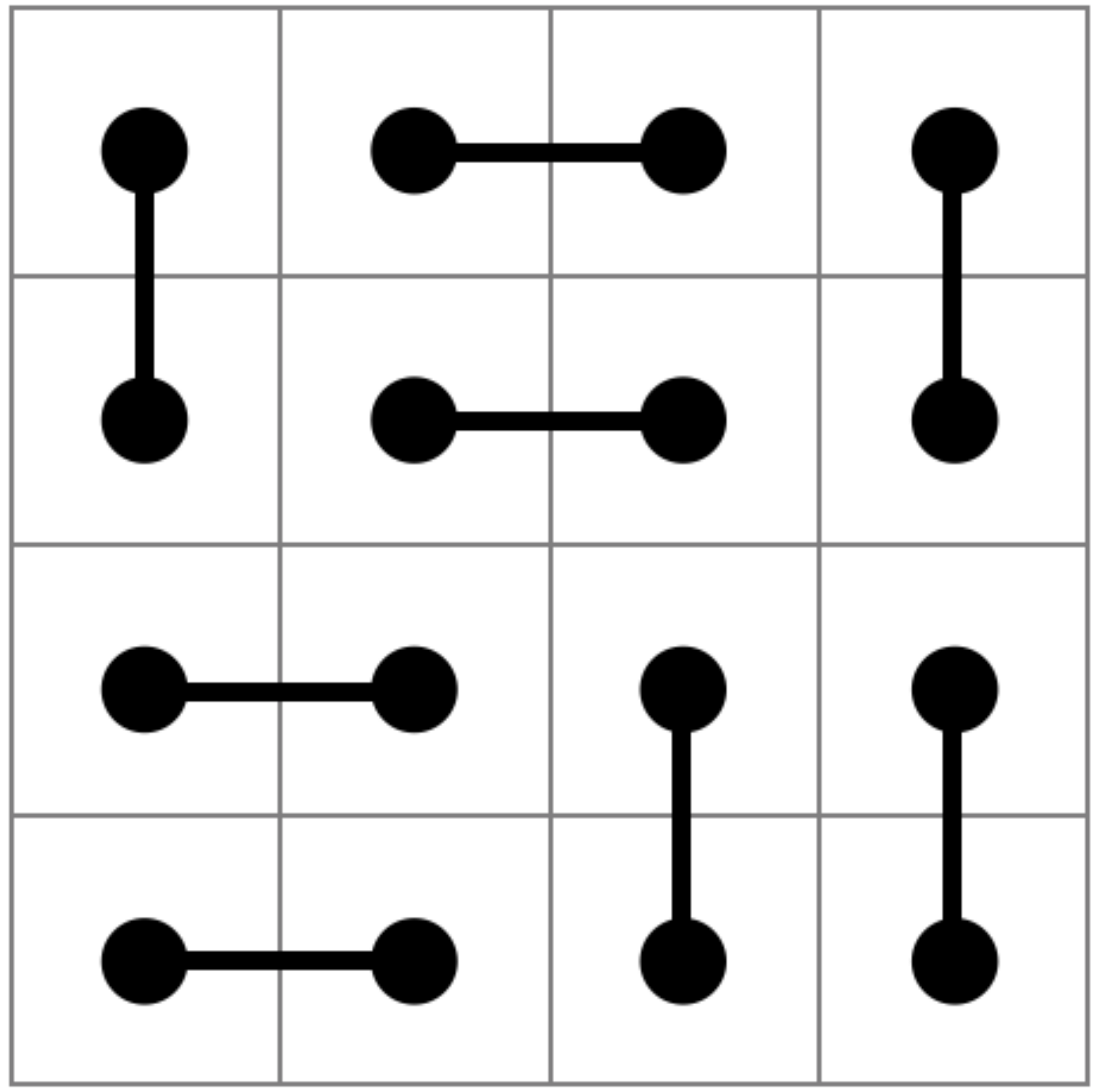
The Grid Graph:

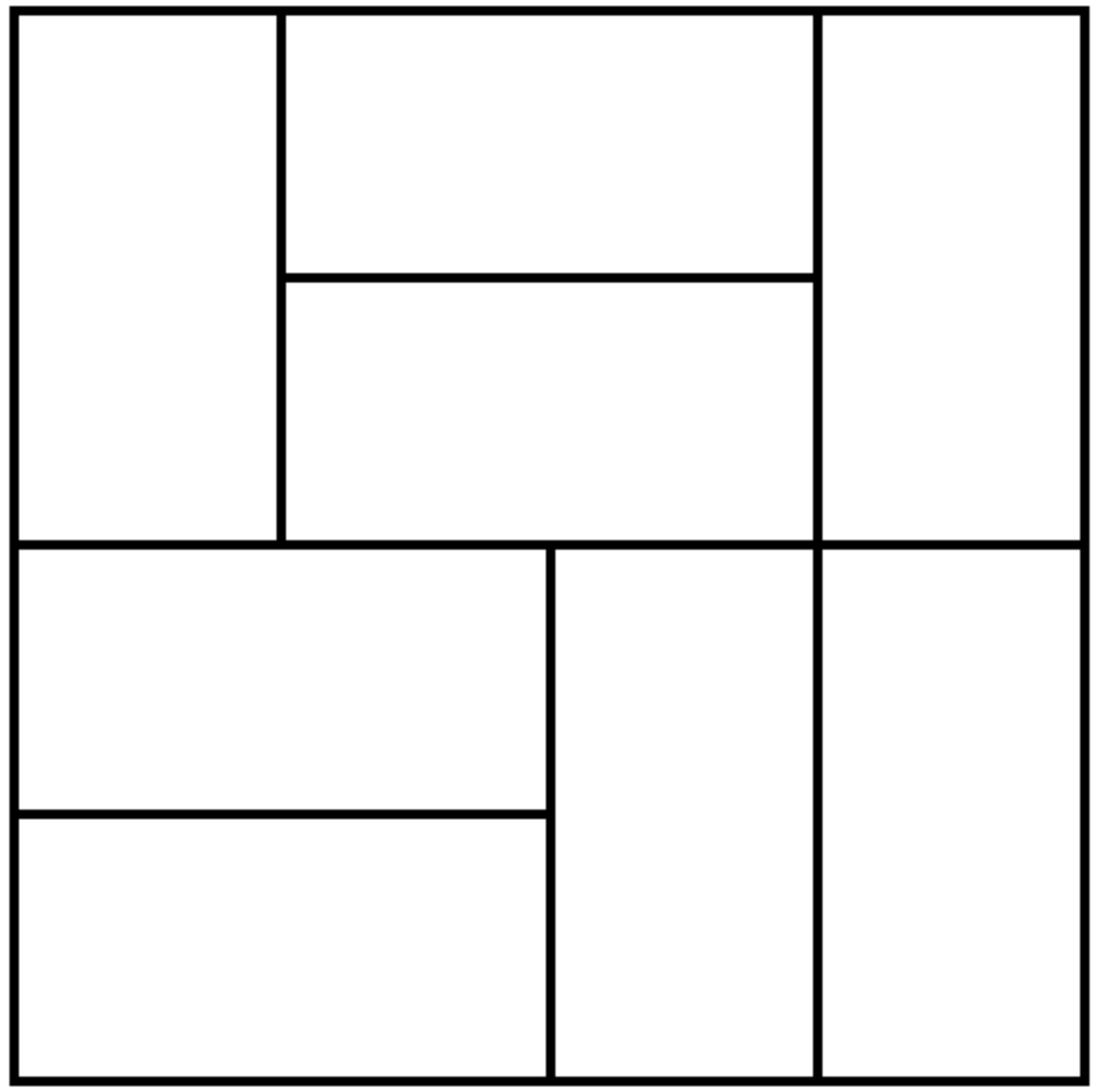




Translating the Problem into Graph Theory

Perfect Matching: A collection of edges in a graph such that every vertex is connected to exactly one edge.

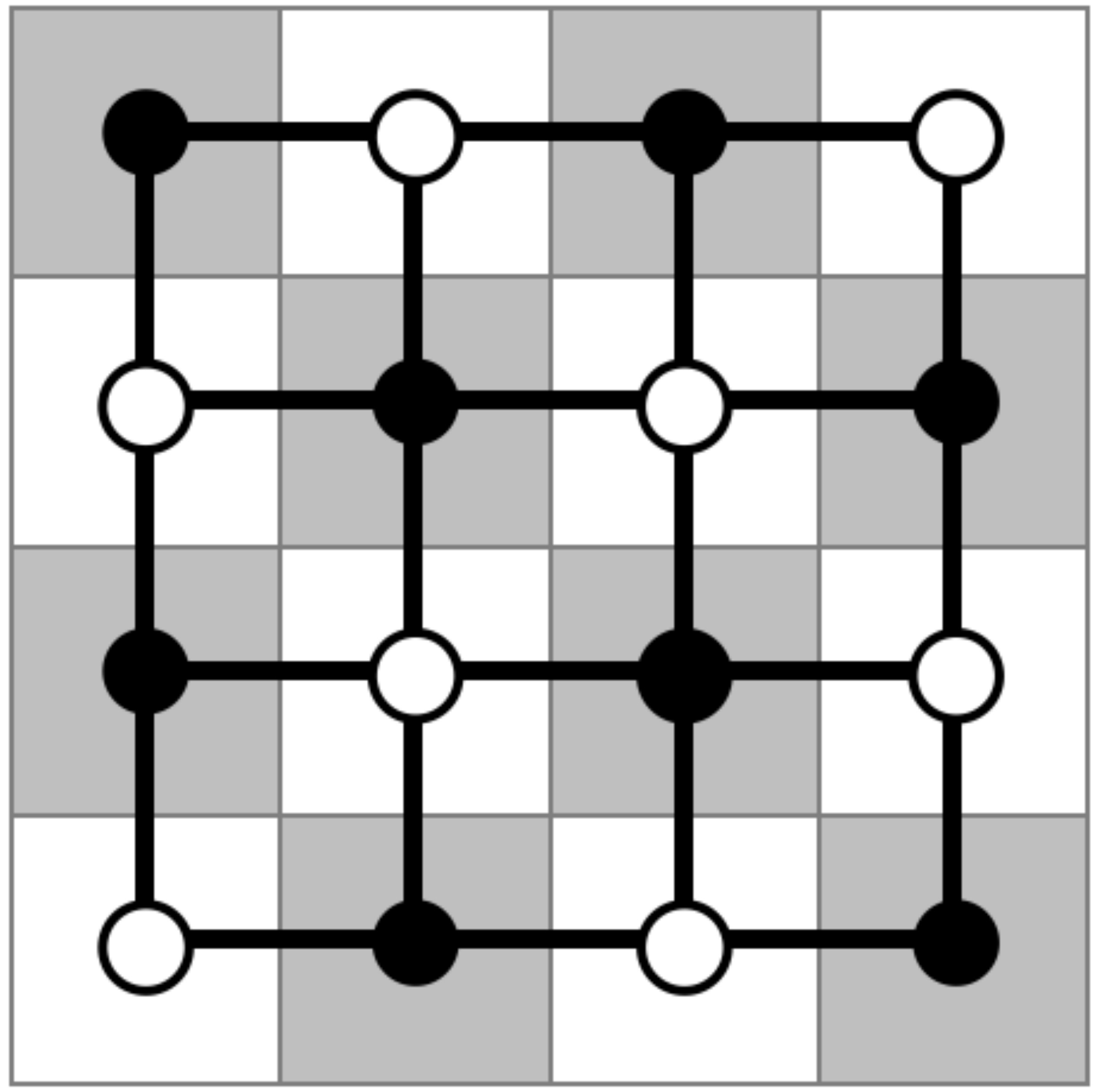




A domino tiling of an n x m grid corresponds to a perfect matching of the n x m grid graph

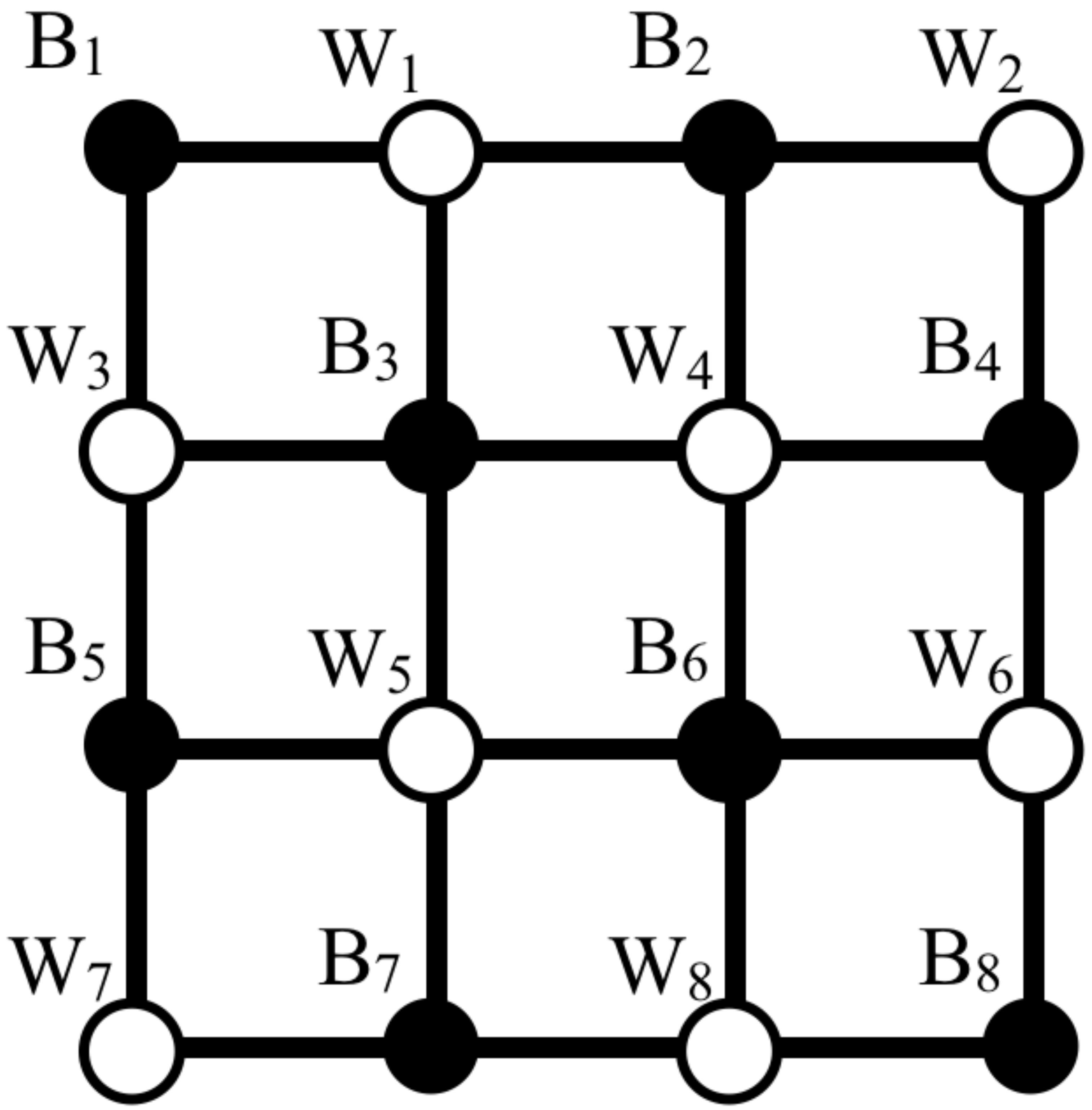
Coloring and Labeling the Vertices

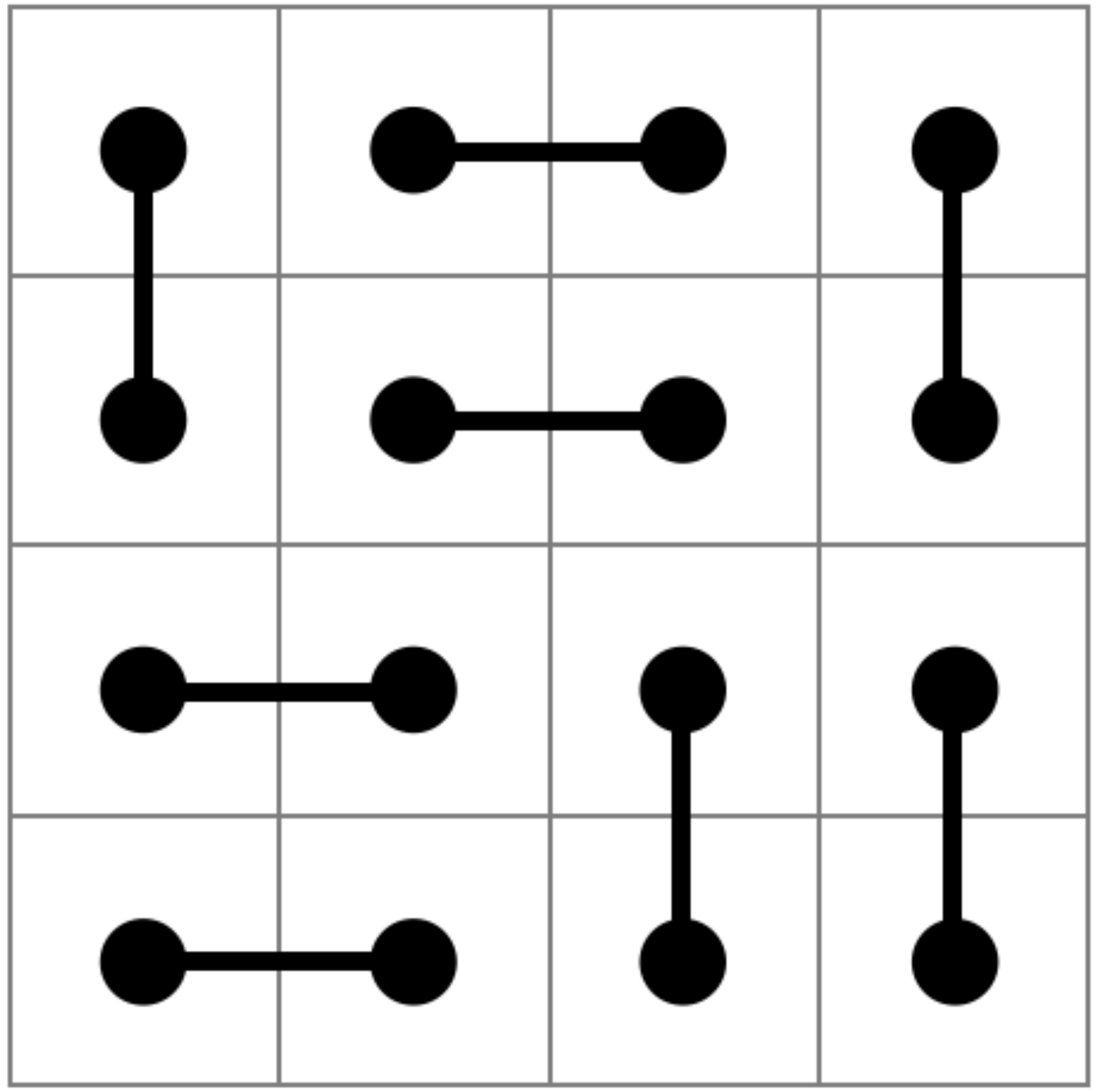
The Grid Graph is Bipartite:



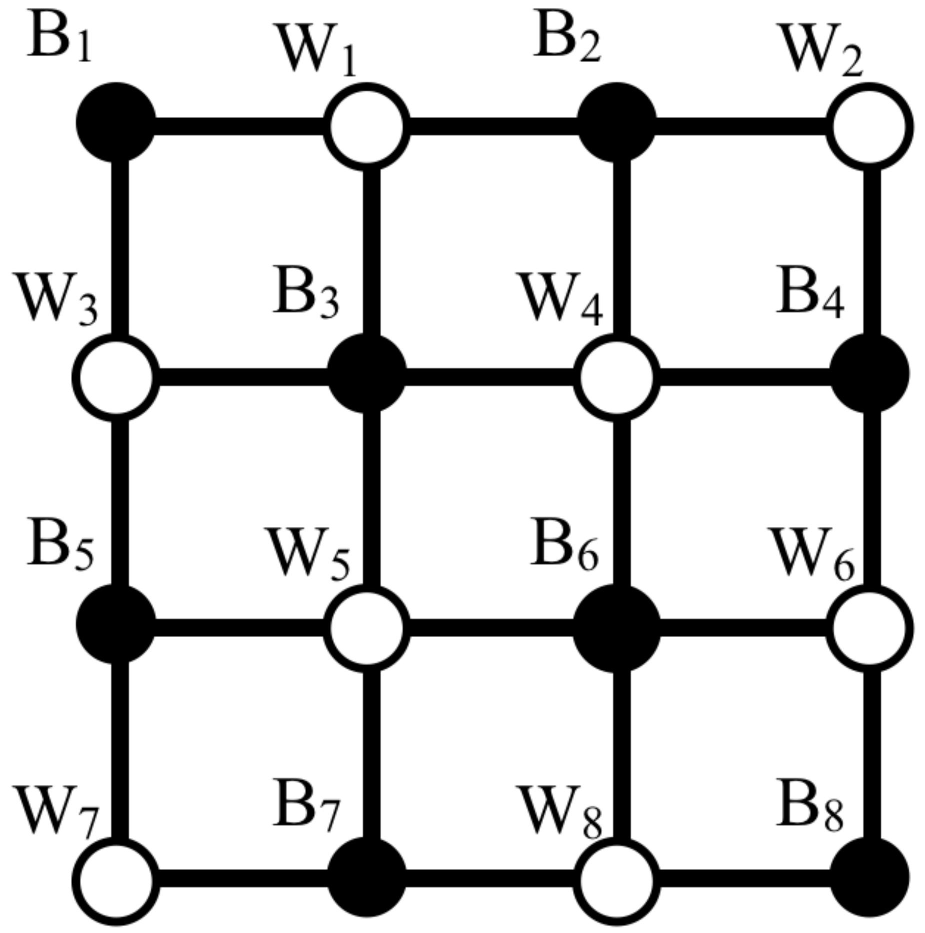
- There are eight black vertices
- There are eight white vertices
- Every edge connects a black vertex to a white vertex

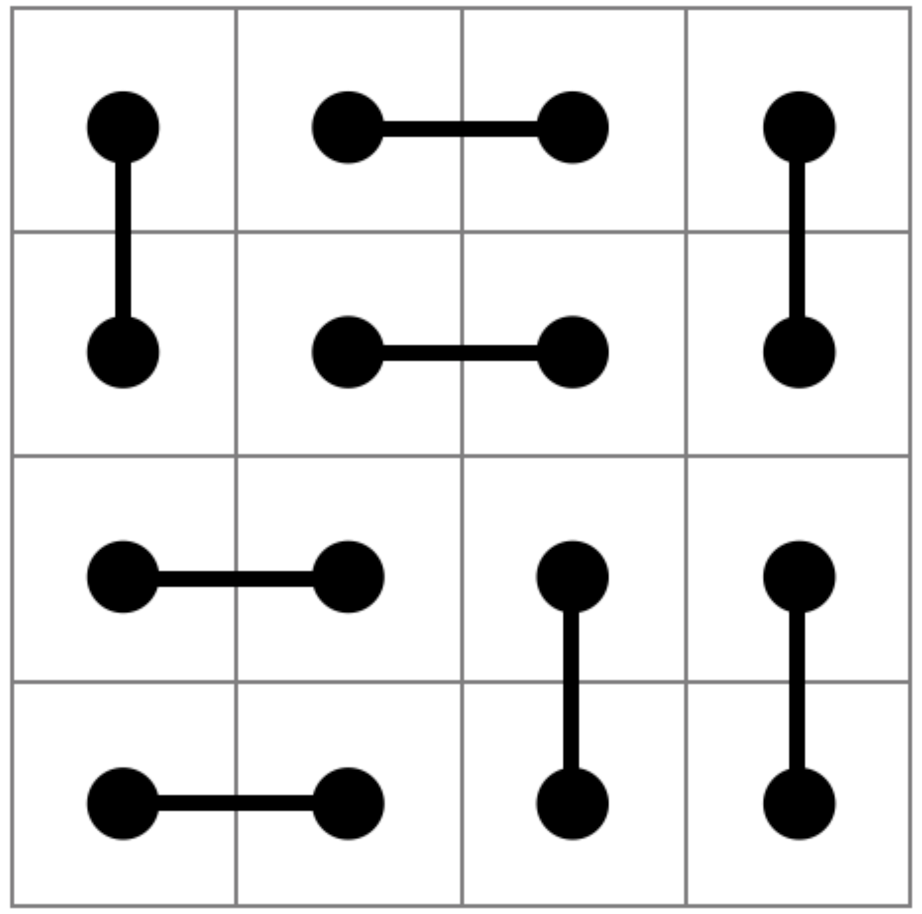
Coloring and Labeling the Vertices

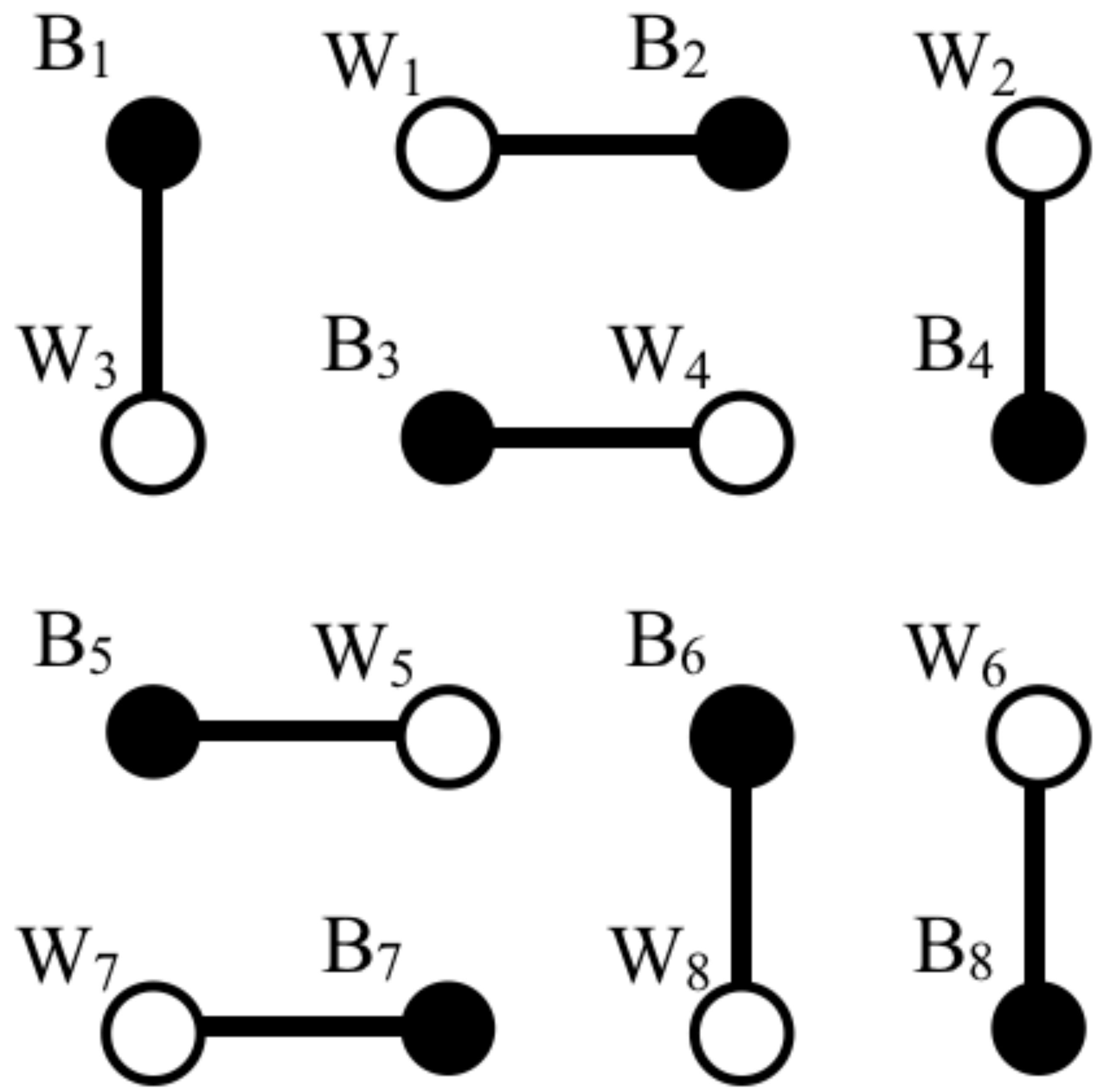




Getting a permutation from a perfect matching





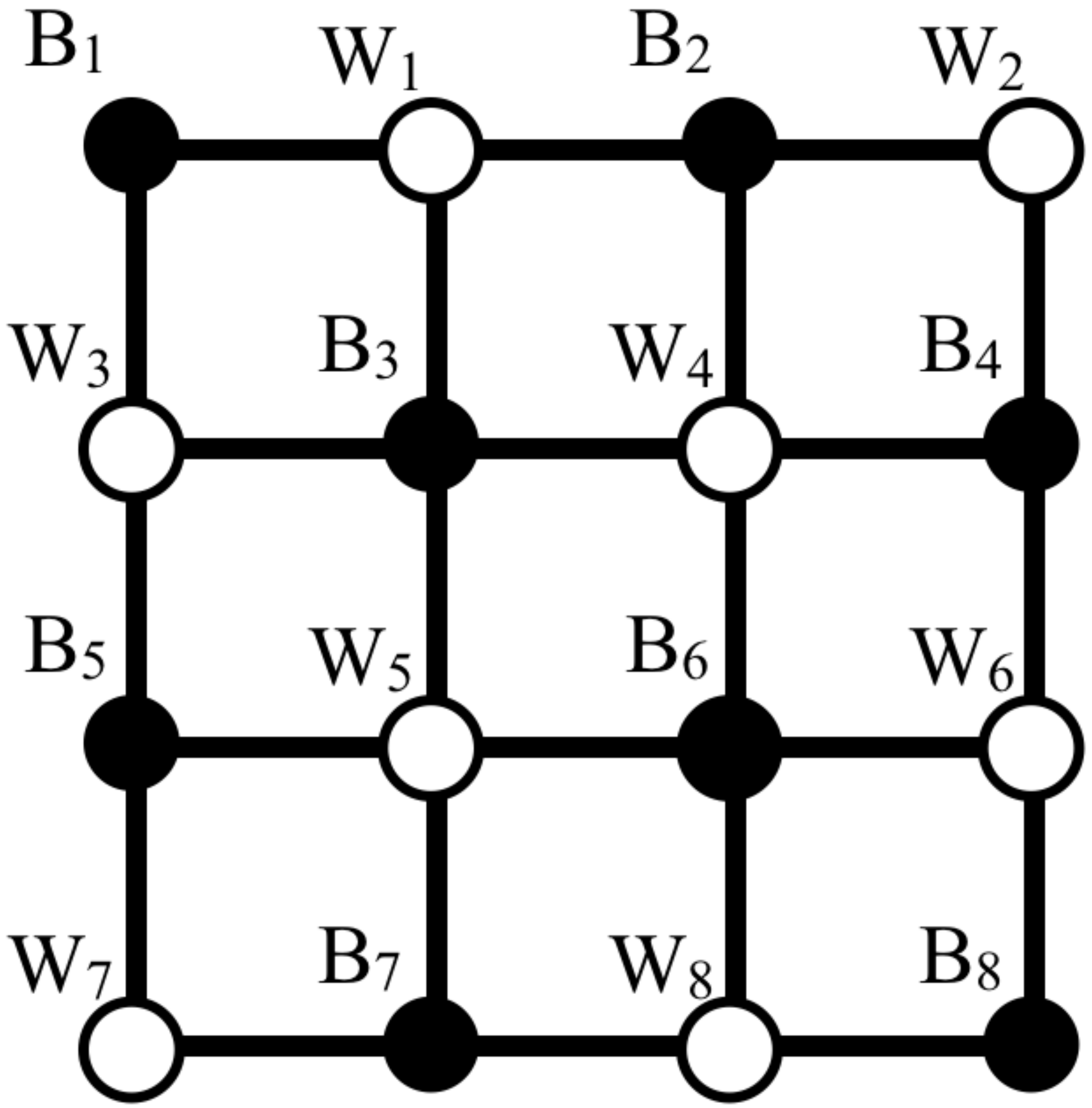


$1 \longmapsto 3$
$2 \longmapsto 1$
$3 \longmapsto 4$
$4 \longmapsto 2$
$5 \longmapsto 5$
$6 \longmapsto 1$
$7 \longmapsto 7$
$8 \longmapsto 6$

Every perfect matching gives rise to a permutation σ defined by

$$\sigma(i)=j \quad \Longleftrightarrow \quad B_i \text{ touches } W_j$$

When does a permutation give rise to a perfect matching?



If σ sends i to j, then B_i must be touching W_j in the grid graph

If for all i, B_i and $W_{\sigma(i)}$ are touching in the grid graph, then σ gives rise to a perfect matching

The Adjacency Matrix

(Move To Chalkboard)

The Adjacency Matrix

$$A_{i,j} = \begin{cases} 1 & B_i \text{ touches } W_j \text{ in the grid graph} \\ 0 & \text{Else} \end{cases}$$

Then, a permutation σ gives rise to a perfect matching if and only if:

$$A_{1,\sigma(1)} \cdot A_{2,\sigma(2)} \cdot A_{3,\sigma(3)} \cdots A_{8,\sigma(8)} = 1$$

A Formula for Perfect Matchings

$$\sum_{\sigma \in S_8} A_{1,\sigma(1)} \cdot A_{2,\sigma(2)} \cdot A_{3,\sigma(3)} \cdots A_{8,\sigma(8)} = T_{4,4}$$

Does this look familiar?

$$\det(A) = \sum_{\sigma \in S_8} \mathrm{sgn}(\sigma) \cdot A_{1,\sigma(1)} \cdot A_{2,\sigma(2)} \cdot A_{3,\sigma(3)} \cdots A_{8,\sigma(8)}$$

The Kasteleyn Weighting

Regular Adjacency Matrix:

$$A_{i,j} = \begin{cases} 1 & B_i \text{ touches } W_j \text{ in the grid graph} \\ 0 & \text{Else} \end{cases}$$

Kasteleyn Adjacency Matrix:

$$A_{i,j} = \begin{cases} i & B_i \text{ touches } W_j \text{ vertically} \\ 1 & B_i \text{ touches } W_j \text{ horizontally} \\ 0 & \text{Else} \end{cases}$$

The Kasteleyn Weighting

With this new weighting:

$$T_{n,m} = |\det(A)|$$

But also by using properties of determinants:

$$|\det(K)| = |\det(-AA^T)| = |\det(A)\det(A^T)| = |\det(A)|^2$$

Therefore:

$$T_{n,m} = |\det(K)|^{1/2}$$

Calculating det(K)

Important fact:

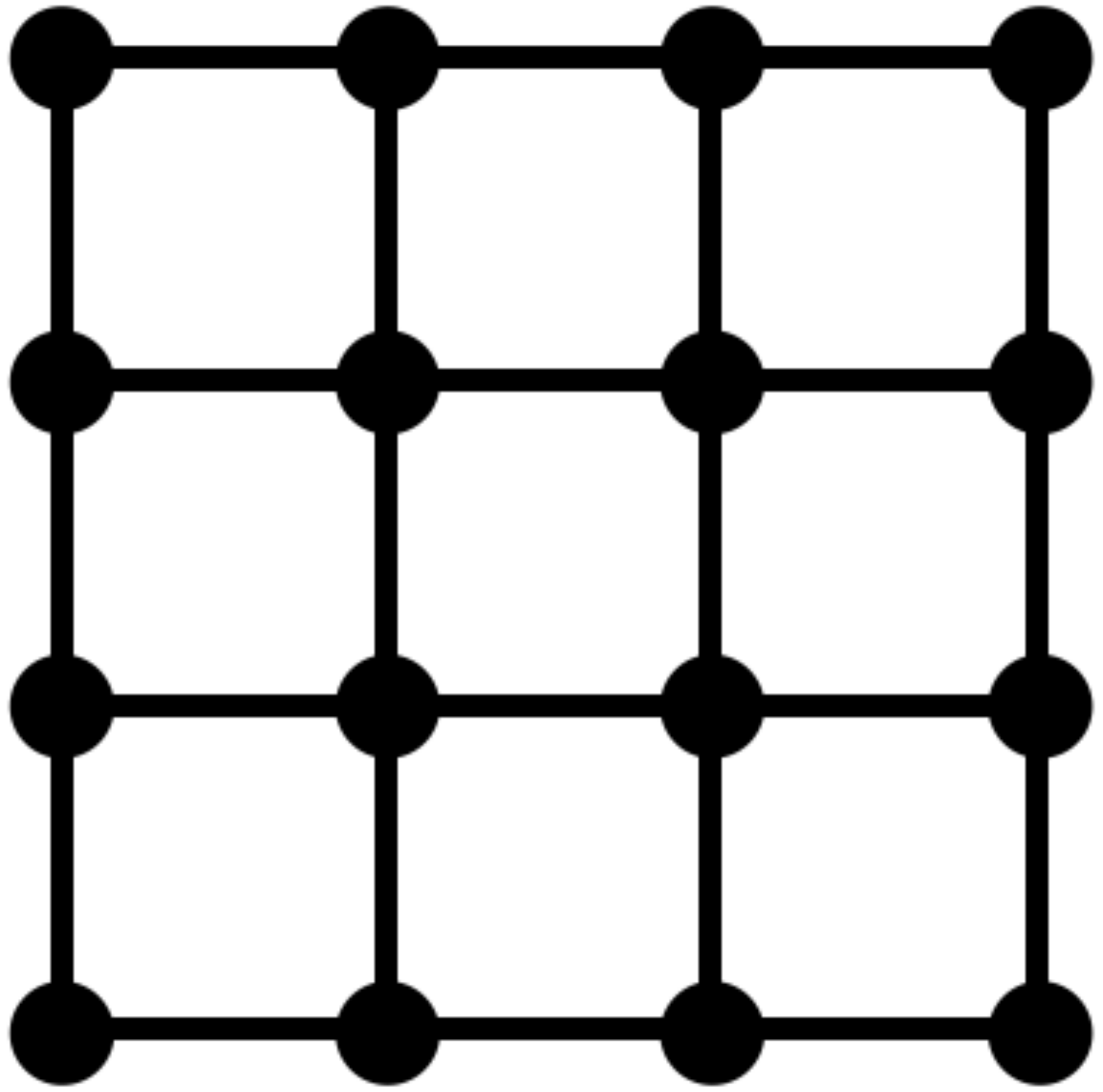
The determinant is the product of the eigenvalues

So what are the eigenvalues of K?

The Cartesian Product of Two Graphs

(Demonstration on Chalkboard)

The Cartesian Product of Two Graphs



$=$

$\square$

(With Kasteleyn Weighting) (With edge weight i) (With edge weight 1)

The Cartesian Product of Two Graphs

Important Theorem:

If λ is an eigenvalue of A, and μ is an eigenvalue for B
Then $\lambda+\mu$ is an eigenvalue for $A \square B$

So in order to determine the eigenvalues of the grid graph,
we just need to find the eigenvalues of the path graphs:

Eigenvalues of the Grid Graph

For a path graph with m vertices and edge weight 1:

$$\lambda_j = 2\cos\left(\frac{\pi j}{m+1}\right)$$

For j = 1,2,…,m

For a path graph with n vertices and edge weight i:

$$\mu_k = 2i\cos\left(\frac{\pi k}{n+1}\right)$$

For k = 1,2,…,n

Determinant of the Grid Graph

Thus, the eigenvalues of the grid graph with Kasteleyn weighting are:

$$2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)$$

j = 1,2,…,m
k = 1,2,…,n

Hence, the determinant of the grid graph is:

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right)$$

Putting It All Together

$T_{n,m} = |\det(K)|^{1/2}$

$$= \left| \prod_{j=1}^{m} \prod_{k=1}^{n} \left(2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right)\right|^{1/2}$$

$$= \prod_{j=1}^{m} \prod_{k=1}^{n} \left| 2\cos\left(\frac{\pi j}{m+1}\right) + 2i\cos\left(\frac{\pi k}{n+1}\right)\right|^{1/2}$$

$$= \prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos\left(\frac{\pi j}{m+1}\right)^2 + 4\cos\left(\frac{\pi k}{n+1}\right)^2\right)^{1/4}$$

The Final Formula

$$T_{n,m} = \prod_{j=1}^{m} \prod_{k=1}^{n} \left(4\cos\left(\frac{\pi j}{m+1}\right)^2 + 4\cos\left(\frac{\pi k}{n+1}\right)^2 \right)^{1/4}$$

Acknowledgements

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- My parents

Bibliography

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